CATEGORIZING DECISION MAKING UNITS BASED ON MULTI-LEVEL FRONTIERS IN DATA ENVELOPMENT ANALYSIS

(Only complete the Author list after the review process has been completed)

ROBABEH ESLAMI
Department of Mathematics, Faculty of Technology and Engineering, South Tehran Branch, Islamic Azad University, Tehran, Iran
sinhx_2002@yahoo.com (Corresponding)

MOHAMMAD KHOVEYNI
Department of Applied Mathematics, Yadegar-e-Imam Khomeini (RAH) Shahre Rey Branch, Islamic Azad University, Tehran, Iran
mohammadkhoveyni@gmail.com

ABSTRACT

This study comments on the data envelopment analysis (DEA) approach which was presented to grade countries/territories by Yang et al. (2016) [Yang, G.-l., Ahlgren, P., Yang, L., Rousseau, R., & Ding, J. (2016). Using multi-level frontiers in DEA models to grade countries/territories. Journal of Informetrics, 10, 238–253]. We show the fundamental problem of their approach and propose a DEA approach to tackle this problem.

Key words: Data envelopment analysis (DEA); Strongly and weakly efficient frontiers; Efficiency

INTRODUCTION

A DEA approach was proposed by Yang, Ahlgren, Yang, Rousseau, and Ding (YAYRD) (2016) for grading countries/territories by using multi-level frontiers in DEA models. However, their approach may not be able to grade countries/territories correctly. In what follows, we describe the drawback of the YAYRD approach.

Consider a set of n homogeneous decision-making units (DMUs), i.e. $\{DMU_j | j = 1,2, \dots, n\}$, where DMU_j uses the input $\mathbf{x}_j = (x_{1j}, \dots, x_{ij}, \dots, x_{mj})^t \underset{\neq}{\geq} \mathbf{0}_m$ to produce the output $\mathbf{y}_j = (y_{1j}, \dots, y_{rj}, \dots, y_{sj})^t \underset{\neq}{\geq} \mathbf{0}_s$. Under the variable returns to scale (VRS) assumption, the production possibility set (PPS) is defined as below (Banker *et al.*, (BCC), 1984):

$$T_V = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \middle| \sum_{j=1}^{n} \lambda_j \mathbf{x}_j \leq \mathbf{x}, \sum_{j=1}^{n} \lambda_j \mathbf{y}_j \geq \mathbf{y}, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0; j = 1, \dots, n \right\}. \qquad (1)$$

The following input-oriented DEA model under the VRS assumption was applied by Cook and Zhu (2005) for specifying the efficiency status of the target DMU, e.g. DMU_o ($o \in \{1,2, \dots, n\}$), (Cook and Zhu, 2005, Chapter 1, p. 10, Table (1-2)):

Input orientation:

$$min \quad \theta - \varepsilon(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+)$$

$$
\begin{aligned}
s.t.\quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io},\ i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro},\ r = 1, \dots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0,\ j = 1, \dots, n,
\end{aligned}
\tag{2}
$$

where ε is a non-Archimedean small positive number.

Definition 1. DMU_o is fully efficient if and only if the following two conditions are satisfied:

- $\theta^* = 1$,
- All slacks equal zero, i.e. $s_i^{-^*} = 0$ (for all i) and $s_r^{+^*} = 0$ (for all r).

Definition 2. DMU_o is weakly efficient if and only if the following two conditions are satisfied:

- $\theta^* = 1$,
- $s_i^{-^*} \neq 0$ and/or $s_r^{+^*} \neq 0$ for some i and r.

Remark 1. As per Definitions 1 and 2, it is deduced that there is no possibility of improving any of the inputs and the outputs of a fully efficient DMU.

Definition 3. Strongly efficient frontier (SEF) is defined as follows:

$$(EF)_{strong} = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in T_V \middle| in\ evaluating \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, \theta^* = 1\ and\ s_i^{-^*} = s_r^{+^*} = 0\ for\ all\ i\ and\ r \right\},$$

Definition 4. Weakly efficient frontier (WEF) is defined as follows:

$$(EF)_{weak} = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in T_V \middle| in\ evaluating \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, (\theta^* = 1\ or\ \varphi^* = 1)\ and\ \left(s_i^{-^*} \neq 0\ and/or\ s_r^{+^*} \neq 0\ for\ some\ i\ and\ r\right) \right\}.$$

Remark 2. As per Definition 3, DMU_o is *fully efficient* (or *BCC-efficient* or *strongly efficient*) if and only if it is on the SEF; also based on Definition 4, DMU_o is *weakly efficient* if and only if it is on the WEF. Otherwise, it is called *inefficient*.

The input efficiency of DMU_o can be determined by applying the input-oriented BCC model as follows (Banker *et al.*, 1984):

$$
\begin{aligned}
& min\quad \theta \\
s.t.\quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io},\ i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro},\ r = 1, \dots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0,\ j = 1, \dots, n,
\end{aligned}
\tag{3}
$$

where $s_i^-\ (i = 1, \dots, m)$ and $s_r^+\ (r = 1, \dots, s)$ represe. Assume that $\left(\theta^*, \boldsymbol{\lambda}^*, s_1^{-^*}, \dots, s_m^{-^*}, s_1^{+^*}, \dots, s_s^{+^*}\right)$ is an optimal solution of model (3).

Definition 5. The *reference set* of DMU_o (E_o) is defined as $E_o = \{j | \lambda_j^* > 0 \ \ in\ all\ optimal\ solutions\ of\ model\ (3), j = 1, \dots, n\}$.

The problem of the YAYRD approach

The strongly efficient frontier is defined by Yang *et al.* (2016) as follows:

Definition 6 (Yang *et al.*, 2016). The efficient frontier of PPS (T_V) is defined as below:

$$EF = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in T_V \middle| \ there\ is\ no\ \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix} \in T_V\ such\ that\ \begin{pmatrix} -\hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix} > \begin{pmatrix} -\mathbf{x} \\ \mathbf{y} \end{pmatrix} \right\}. \quad (4)$$

Yang *et al.* (2016) applied model (3) and Definition 6 for grading DMUs into different levels. They mentioned that if $\theta^* = 1$ then DMU_o is on the SEF. This statement may not be correct because, as per Definitions 1 and 2, if $\theta^* = 1$ then DMU_o is not necessarily on the SEF. Note that, in this case, DMU_o may be on the WEF. Hence, the following theorem, presented by Yang *et al.* (2016), is not correct. This is the fundamental problem of the YAYRD approach.

Theorem 1 (Yang *et al.*, 2016, p. 244). *There is no intersection between any two efficient frontier grades.*

In what follows, we present a numerical example for illustrating the problem of the YAYRD approach.

Consider a set of nine hypothetical DMUs with one input and one output, as presented in Table 1. Fig. 1 shows the multi-level SEFs of the PPS corresponding to the given DMUs.

Table 1. DMUs' data.

DMU	A	B	C	D	E	F	G	H	I
Input	2	2	3	8	14	4	9	6	10
Output	2	4	8	10	10	6	9	3	7

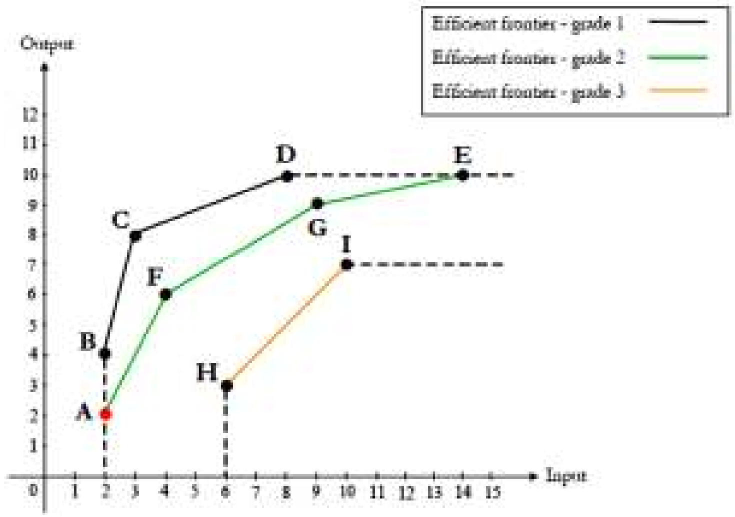

Fig. 1. Multi-level strongly efficient frontiers.

As represented in Fig. 1, as per Definition 6, the piecewise lines BCD, $AFGE$, and HI are the efficient frontiers for performance grades 1, 2, and 3, respectively. The obtained results from grading the DMUs have been listed in Table 2. As seen in this table, DMU_A belongs to grade 2.

Table 2. The performance grades of the DMUs.

DMU	A	B	C	D	E	F	G	H	I
Grade 1	---	✓	✓	✓	---	---	---	---	---
Grade 2	✓	---	---	---	✓	✓	---	---	---
Grade 3	---	---	---	---	---	---	---	✓	✓

Here, we are going to grade these DMUs by using model (3). The obtained results by using model (3) have been shown in Table 3. According to Table 3, DMU_A belongs to grade 1 while, as per in Table 2, this DMU belongs to grade 2. So, DMU_A is the intersection between grades 1 and 2. Hence, Theorem 1 is not correct. This is the problem of the YAYRD approach because their approach states DMU_o is on the SEF if $\theta^* = 1$ while, in this case, DMU_o may be on the WEF like DMU_A in the above example.

Table 3. The input efficiency score by solving model (3).

DMU	A	B	C	D	E	F	G	H	I
Grade 1	1.0000	1.0000	1.0000	1.0000	0.5714	0.6250	0.6111	0.3333	0.2750
Grade 2	---	---	---	---	1.0000	1.0000	1.0000	0.6667	0.5667
Grade 3	---	---	---	---	---	---	---	1.0000	1.0000

Methodology

To tackle the problem of the YAYRD approach, we first determine the efficiency status of DMUs by applying the Additive model as below:

$$
\begin{aligned}
z^* = max\ & \textstyle\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
\text{s.t. } & \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io},\ \ i = 1, \dots, m, \\
& \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro},\ \ r = 1, \dots, s, \\
& \textstyle\sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0,\ \ j = 1, \dots, n, \\
& s_i^- \geq 0,\ \ i = 1, \dots, m, \\
& s_r^+ \geq 0,\ \ r = 1, \dots, s.
\end{aligned}
\tag{5}
$$

Definition 7. DMU_o is called *additive-efficient* if and only if $z^* = 0$ where z^* is the optimal objective value of model (5).

Theorem 2 (Cooper *et al.*, 2007, Chapter 4). *DMU_o is additive-efficient if and only if it is strongly efficient.*

Proof. See Chapter 4 from Cooper *et al.* (2007). **Q.E.D.**

Second, we introduce the obtained strongly efficient DMUs from model (5) as the first group. These DMUs are on the efficient frontier-grade 1. Then, by eliminating these strongly efficient DMUs from the reference set and resolving model (5) for the remaining DMUs, we attain the new strongly efficient DMUs that are on the efficient frontier-grade 2 as the second group. Similarly, we repeat this grading process until there is no DMU left. Clearly, after finishing this process, there is no intersection between any two groups.

Here, our proposed approach is used for grading the given DMUs in Table 1. Based on Table 4, DMU_B, DMU_C, and DMU_D belong to grade 1; DMU_A, DMU_E, DMU_F, and DMU_G belong to grade 2; and DMU_H and DMU_I belong to grade 3. Therefore, each DMU only belongs to one group and there is no intersection among groups 1, 2, and 3.

Table 4. Results of model (5) into different levels.

DMU	A	B	C	D	E	F	G	H	I
Grade 1	2	0	0	0	6	3	3.5	8	8
Grade 2	0	---	---	---	0	0	0	5	4.33
Grade 3	---	---	---	---	---	---	---	0	0

Conclusions

Yang *et al.* (2016) proposed a DEA-based approach for grading DMUs. However, their approach may incapable of grading DMUs correctly. So, in this study, we present a DEA-based approach to tackle the problem of the YAYRD approach.

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